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# Decomposition and Uncoupling of Multi-Degree-of-Freedom Gyroscopic Conservative Systems

**Ranislav M. Bulatovic**
Faculty of Mechanical Engineering,
University of Montenegro,
Dzordza Vashingtona bb,
81000 Podgorica, Montenegro
e-mail: ranislav@ucg.ac.me

**Firdaus E. Udwadia**[1]
Departments of Aerospace and Mechanical
Engineering, Civil and Environmental Engineering,
and Information and Operations Management,
University of Southern California,
Los Angeles, CA 90089
e-mail: fudwadia@usc.edu

*This paper explores the decomposition of linear, multi-degree-of-freedom, conservative gyroscopic dynamical systems into uncoupled subsystems through the use of real congruences. Two conditions, both of which are necessary and sufficient, are provided for the existence of a real linear coordinate transformation that uncouples the dynamical system into independent canonical subsystems, each subsystem having no more than two-degrees-of-freedom. New insights and conceptual simplifications of the behavior of such systems are provided when these conditions are satisfied, thereby improving our understanding of their complex dynamical behavior. Several analytical results useful in science and engineering are obtained as consequences of these twin conditions. Many of the analytical results are illustrated by several numerical examples to show their immediate applicability to naturally occurring and engineered systems.* [DOI: 10.1115/1.4063504]

*Keywords: linear gyroscopic potential system, uncoupling into at most two-degree-of-freedom subsystems, necessary and sufficient conditions, congruence and orthogonal congruence transformations, commutation condition, perfectly matched system, dynamics, structures, vibration*

## 1 Introduction

Understanding the fundamental behavior of large-scale linear gyroscopic potential dynamical systems has, in recent years, become an important topic of research, especially in aerospace and mechanical engineering systems, because of the prevalence of gyroscopic forces in, for instance, the rotary motion in rotating flexible machinery, spinning elastic systems, astrodynamics, satellite control, problems related to the motion of charged systems in magnetic fields, order reduction of systems with symmetries, and when using rotating frames of reference in analytical dynamics.

In this paper, we consider a multi-degree-of-freedom potential (conservative) system subjected to gyroscopic forces described by

$$\tilde{M}\ddot{q} + \tilde{G}\dot{q} + \tilde{K}q = 0 \tag{1}$$

where $\tilde{M}$, $\tilde{G}$, and $\tilde{K}$ are $n$-by-$n$ constant real matrices; $\tilde{M}$ and $\tilde{K}$ are symmetric, and $\tilde{G}$ is skew-symmetric ($\tilde{G} = -\tilde{G}^T$). The $n$-vector of generalized coordinates is denoted by $q$, and the dots indicate differentiation with respect to time $t$. We aim to improve our analytical and intuitive understanding of the fundamental dynamics of such systems.

Equation (1) represents a set of coupled second-order ordinary-differential equations and can be obtained by the application of Lagrange's equations with the Lagrangian [1]

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T\tilde{M}\dot{q} + \frac{1}{2}\dot{q}^T\tilde{G}q - \frac{1}{2}q^T\tilde{K}q \tag{2}$$

Consider the real congruent nonsingular transformation $q = Pp$ so that

$$\tilde{M} \to M = P^T\tilde{M}P,\ \tilde{K} \to K = P^T\tilde{K}P,\ \tilde{G} \to G = P^T\tilde{G}P \tag{3}$$

When the matrix $\tilde{M}$ is positive definite and $\tilde{G} = 0$ (pure potential system or conservative non-gyroscopic system), one can always find the real transformation matrix $P$ so that the new inertia and potential (stiffness) matrices are diagonal, i.e., in new coordinates $p$, called normal (principal or modal) coordinates, the system is decomposed into an independent (uncoupled) single-degree-of-freedom subsystems. This classical result was established by Weierstrass in 1858 in the context of the simultaneous reduction of two quadratic forms to sums of squares [2]. When $\tilde{G} \neq 0$, the system is not completely decomposable because of the coordinate change in Eq. (3) that makes $M$ and $K$ diagonal retains $G$ as a skew-symmetric matrix. We note that the minimum number of degrees-of-freedom necessary to incorporate gyroscopic effects is two and that the eigenvalues of $G$ are conjugate purely imaginary pairs and zeros. Therefore, it is natural to ask whether or not we can decompose system (1) into independent subsystems, each of which has no more than two-degrees-of-freedom, by means of a real change of coordinates conferred by using a real congruence transformation. The answer to this question is the subject of this paper.

The intent of this paper is to show that multi-degree-of-freedom gyroscopic potential systems can be uncoupled when certain

[1]Corresponding author.

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necessary and sufficient conditions are satisfied, much like damped potential systems can be when they satisfy the necessary and sufficient condition first proposed by Caughey–O'Kelly [3]. And just as the Caughey–O'Kelly condition leads to a drastic simplification and therefore a much-improved understanding of the dynamics of multi-degree-of-freedom damped potential systems in terms of the behavior of simple, uncoupled, canonical one-degree-of-freedom subsystems, the conditions obtained in this paper lead to a substantial simplification and improvement in our understanding of the behavior of multi-degree-of-freedom gyroscopic potential systems in terms of simple, two-degrees-of-freedom uncoupled, canonical gyroscopic potential subsystems, and simple, one-degree-of-freedom uncoupled, purely potential subsystems.

We begin with a short prelude in Sec. 2 that presents two algebraic results, the first of which is well known; the second is perhaps not as widely known and is perhaps limited to a smaller audience of researchers. Both these results are basic to our further development. Section 3 considers systems in which the mass matrix $\tilde{M} > 0$ and the potential (stiffness) matrix $\tilde{K}$ are symmetric. Several new results are formulated, proved, and discussed. Numerical examples are also provided showing the simplicity and efficacy of the analytical constructs developed. Section 4 adduces results from those developed in Sec. 3 for systems in which the mass matrix $\tilde{M}$ is symmetric and the potential matrix $\tilde{K} > 0$. Section 5 provides a summary of our main analytical findings.

## 2 Mathematical Preliminaries

We begin with some useful properties of real skew-symmetric matrices [4].

Lemma 1. *Let $G \neq 0$ be an $n \times n$ real skew-symmetric matrix. Then:*

(a) *$rank(G) = 2m \leq n$ is even.*
(b) *$G$ has a zero eigenvalue of multiplicity $n-2m$, and $2m$ pure imaginary eigenvalues in pairs $\pm i\beta_j$, $i = \sqrt{-1}$, $j = 1, \ldots, m$, which are all simple or semi-simple.*
(c) *there exists a real orthogonal matrix $Q$ such that.*

$$Q^T G Q = \text{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right) \quad (4)$$

*where $\beta_j$, $j = 1, \ldots, m$, are nonzero real numbers.*

The block-diagonal matrix shown in Eq. (4) is the canonical (simplest possible) form of a skew-symmetric matrix with respect to an orthogonal congruence in which the matrix $Q$ is real and orthogonal, while the canonical form for a real symmetric matrix is, of course, a diagonal matrix consisting of its real eigenvalues along the diagonal.

The following assertion plays a central role in all our further considerations. It is a counterpart of the well-known result which states that the necessary and sufficient condition for two real symmetric matrices to be simultaneously diagonalized by a real orthogonal transformation is that the matrices commute in multiplication [4].

Lemma 2. *Let $K = K^T$ and $G = -G^T \neq 0$ be $n$ by $n$ real matrices, and let be $rank(G) = 2m \leq n$. The necessary and sufficient conditions that there exists a real orthogonal matrix $Q$ such that [5]*

$$Q^T K Q = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \quad (5)$$

*and*

$$Q^T G Q = \Gamma = \text{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right) \quad (6)$$

*where all the $\lambda_j$'s are real numbers and all the $\beta_j$'s are nonzero real numbers, are that.*

$$KG^2 = G^2 K \quad (7)$$

and

$$(KG)^2 = (GK)^2 \quad (8)$$

It is clear that the condition given in Eq. (7) implies symmetry of the matrix $KG^2$ since

$$(KG^2)^T = (KGG)^T = G^T G^T K^T = GGK = G^2 K = KG^2 \quad (9)$$

with the last equality following from Eq. (7). Also, when $KG^2$ is symmetric, Eq. (7) follows. In a similar manner, $(KG)^2$ can be shown to be symmetric, and when $(KG)^2$ is symmetric, the condition given in Eq. (8) follows.

The above result, published in a small circulation journal, is little known, and its proof is given in the Appendix.

## 3 Decomposition and Uncoupling of Gyroscopic Systems With Symmetric Stiffness (Potential) Matrices

In this section, we consider multi-degree-of-freedom gyroscopic systems whose mass matrices, $\tilde{M}$, are positive definite ($\tilde{M} = \tilde{M}^T > 0$) and whose potential matrices, $\tilde{K}$, are symmetric, possibly positive semi-definite as when rigid-body motion is included.

Since the matrix $\tilde{M}$ is positive definite, upon premultiplication by $\tilde{M}^{-1/2}$, Eq. (1) can also be written as

$$\ddot{x} + G\dot{x} + Kx = 0 \quad (10)$$

where the symmetric matrix

$$K = \tilde{M}^{-1/2} \tilde{K} \tilde{M}^{-1/2} \quad (11)$$

the skew-symmetric matrix

$$G = \tilde{M}^{-1/2} \tilde{G} \tilde{M}^{-1/2} \quad (12)$$

and $q = \tilde{M}^{-1/2} x$. In this section, we shall be mostly dealing with the system described by Eq. (10), which is equivalent to the one described in Eq. (1). We assume that the matrix $G$ in the system described by Eq. (10) has rank $2m \leq n$. Our aim is to transform the system described in Eq. (10) into a simpler set of independent, uncoupled subsystems. Our first result states that this is possible through the use of a real orthogonal congruence transformation.

**Result 1**

Consider the system described in Eq. (10) in which the skew-symmetric matrix has rank $2m \leq n$. Then, the necessary and sufficient conditions for Eq. (10) to be decomposed by an orthogonal congruence transformation into uncoupled subsystems, $m$ of which are two-degrees-of-freedom and $n - 2m$ of which are single-degree-of-freedom subsystems, is that

$$KG^2 = G^2 K \quad (13)$$

and

$$(KG)^2 = (GK)^2 \quad (14)$$

The uncoupled equations in the principal coordinates $p$ have the following form.

$$\ddot{p} + \Gamma \dot{p} + \Lambda p = 0 \quad (15)$$

where,

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad (16)$$

and

$$\Gamma = \text{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right) \quad (17)$$

*Proof.* Using the real orthogonal transformation $x = Qp$ with $QQ^T = I_n$, Eq. (10) becomes

$$Q\ddot{p} + GQ\dot{p} + KQp = 0$$

Premultiplication on both sides by $Q^T$ results in the equation

$$\ddot{p} + Q^T GQ\dot{p} + Q^T KQp = 0 \tag{18}$$

in the new coordinate $p$.

Now, according to Lemma 2, the conditions in Eqs. (13) and (14) are both *necessary and sufficient* for the existence of a real orthogonal matrix $Q$ such that $Q^T GQ = \Gamma$ and $Q^T KQ = \Lambda$ so that Eq. (18) becomes

$$\ddot{p} + \Gamma\dot{p} + \Lambda p = 0$$

were,

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

and

$$\Gamma = \mathrm{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right)$$

■

The matrix $\Lambda$ contains the eigenvalues of the matrix $K$, and the matrix $\Gamma$ indicates that the nonzero eigenvalues of $G$ are $\pm i\beta_j$, $j = 1, 2, \ldots, m$. We shall refer to $p$ as the principal coordinate.

In this paper, we shall refer to an orthogonal matrix $Q$ that makes $Q^T KQ = \Lambda$ and $Q^T GQ = \Gamma$ where $\Lambda$ and $\Gamma$ are described in Eqs. (16) and (17), as "*the matrix Q that simultaneously quasi-diagonalizes (or leads to the simultaneous quasi-diagonalization of) a symmetric n-by-n matrix (like K) and an anti-symmetric n-by-n matrix (like G).*"

*Remark 1.* The matrix $\Gamma^2 = Q^T G^2 Q$ is a diagonal matrix and contains the eigenvalues of $G^2$

$$\Gamma^2 = \mathrm{diag}(-\beta_1^2, -\beta_1^2, \ldots, -\beta_m^2, -\beta_m^2, 0, \ldots, 0) \tag{19}$$

Since $\Lambda$ and $\Gamma^2$ are diagonal,

$$\Gamma^2 \Lambda = \Lambda \Gamma^2 \tag{20}$$

■

As mentioned earlier, the system described by Eq. (1) is equivalent to the system described by Eq. (10); this leads to the following result.

**Result 2**

Assume that the $n$-by-$n$ matrix $\tilde{G}$ of the system described in Eq. (1) has rank $2m \le n$. Then, the necessary and sufficient condition that the system described by Eq. (1) can be transformed by a linear change of coordinates to the one given in Eqs. (15)–(17) is that

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K} \tag{21}$$

and

$$(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})^2 = (\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2 \tag{22}$$

*Proof.* Noting Eqs. (11) and (12), we find that the condition given in Eq. (13) becomes

$$\tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1/2} = \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1/2}$$

Premultiplication and postmultiplication of this equation by $\tilde{M}^{1/2}$ yields Eq. (21).

Similarly, the condition given in Eq. (14) becomes

$$\tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1/2}$$
$$= \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1/2}$$

Upon premultiplication on both sides by $\tilde{M}^{1/2}$ and postmultiplication on both sides by $\tilde{M}^{-1/2}$, we get

$$(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}) = (\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})(\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})$$

which is Eq. (22). ■

*Remark 2.* The real numbers $\lambda_j$ ($j = 1, 2, \ldots, n$), which are the eigenvalues of the matrix $K$, are also the eigenvalues of $\tilde{M}^{-1}\tilde{K}$, while the nonzero eigenvalues $G$, which are $\pm i\beta_j$ ($j = 1, \ldots, m$), are also the nonzero eigenvalues of $\tilde{M}^{-1}\tilde{G}$. ■

*Remark 3.* The necessary and sufficient conditions given in Eqs. (21) and (22) guarantee that the system described by Eq. (1) can be expressed by the uncoupled subsystems shown in Eqs. (15)–(17) through a suitable linear coordinate transformation. The real linear coordinate transformation $q = \tilde{M}^{-1/2}x = \tilde{M}^{-1/2}Qp$ accomplishes this. Note that the response, $q(t)$, of the system described by Eq. (1), will, in general, couple the response of the uncoupled subsystems. ■

Observe that the decoupling conditions Eqs. (21) and (22) (or Eqs. (13) and (14)) hold trivially when either $\tilde{K}(K) = 0$ or $\tilde{G}(G) = 0$. In the first case, the system described in Eq. (15) reduces to the form $\ddot{p} + \Gamma\dot{p} = 0$, while in the second case, as is well-known, the system transforms to the completely uncoupled form $\ddot{p} + \Lambda p = 0$.

A gyroscopic system is said to be perfectly matched if the gyroscopic matrix determined in normal coordinates (modal gyroscopic matrix) contains one and only one nonzero element in each row and column. The concept of perfect matching was introduced in Ref. [6] in the study of the stability of gyroscopic systems (see also Refs. [7,8]). By a permutation of normal coordinates, a perfectly matched system can be written in the form shown in Eqs. (15)–(17) where, clearly, $2m = n$. Then, it follows from Result 2 that in the general case of a gyroscopic system that satisfies conditions Eqs. (21) and (22), the system, by a congruence transformation, splits into a $2m$-dimensional ($2m = rank\tilde{G}$) perfectly matched gyroscopic subsystem and an ($n$–$2m$)-dimensional pure potential subsystem. In particular, if the gyroscopic system is nonsingular (i.e., $\det \tilde{G} \neq 0$, and then $n$ is necessarily even), then the system is perfectly matched if and only if the conditions in Eqs. (21) and (22) hold. On the other hand, as pointed out in Ref. [7], there is a parallel between perfectly matched gyroscopic systems and classically damped non-gyroscopic systems.

A damped system can be (formally) obtained by replacing the gyroscopic matrix $\tilde{G}$ with the symmetric damping matrix $\tilde{D}$ in Eq. (1). Such a system is said to be classically damped if it can be uncoupled by modal analysis into a set of $n$ independent single-degree-of-freedom systems. According to the well-known Caughey–O'Kelly result [3], the single necessary and sufficient condition under which a damped linear system is classically damped is

$$\tilde{K}\tilde{M}^{-1}\tilde{D} = \tilde{D}\tilde{M}^{-1}\tilde{K} \tag{23}$$

The case of classical damping is often assumed in scientific and engineering literature, and more complicated non-classically damped systems are treated as perturbations, small or finite, about a classically damped system (see Ref. [9], for example).

In the case of gyroscopic systems, a similar approach can be developed where the role of the unperturbed system is played by a perfectly matched system [7]. Therefore, from this point of view, Results 1 and 2 can be interpreted as a counterpart of the result of Caughey–O'Kelly [3].

It should be noted that by replacing the gyroscopic matrix $G$ with the symmetric damping matrix $D$ in Eq. (10), the Caughey–O'Kelly necessary and sufficient condition for a system to be classically damped is $KD = DK$; this results from the substitution (see Eqs. (11) and (12)) of $\tilde{D} = M^{1/2}DM^{1/2}$ and $\tilde{K} = M^{1/2}KM^{1/2}$ in Eq. (23). The obvious analog of this commutation condition for our corresponding gyroscopic potential system is simply the single condition $KG = GK$.

The following real-world example with four degrees-of-freedom illustrates the applicability of the result in Result 1.

*Example 1.* Consider a simplified model of two discs mounted on a circular weightless elastic shaft rotating with a constant angular velocity. The system is described by Eq. (10) with

$$G = \xi \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{24}$$

and

$$K = \begin{bmatrix} 2-\xi^2 & 1 & 0 & 0 \\ 1 & 4-\xi^2 & 0 & 0 \\ 0 & 0 & 2-\xi^2 & 1 \\ 0 & 0 & 1 & 4-\xi^2 \end{bmatrix} \tag{25}$$

where $\xi$ represents the angular velocity [10,11].

The matrices in Eqs. (24) and (25) satisfy condition (Eq. (13)) because $G^2 = -\xi^2 I$. Next, we calculate

$$KG = \xi \begin{bmatrix} 0 & 0 & \xi^2-2 & -1 \\ 0 & 0 & -1 & \xi^2-4 \\ 2-\xi^2 & 1 & 0 & 0 \\ 1 & 4-\xi^2 & 0 & 0 \end{bmatrix}$$

This matrix is skew-symmetric so that the matrix $(KG)^2$ is symmetric, i.e., the condition in Eq. (14) is also satisfied. Since *rank(G)* = 4, the system can be transformed by a real orthogonal congruence transformation into two uncoupled two-dimensional subsystems.

Indeed, one easily verifies that the coordinate transformation $x = Qp$, where the columns of the transformation matrix $Q$ are the following orthonormal eigenvectors of the matrix (Eq. (25))

$$q_1 = \frac{\sqrt{2+\sqrt{2}}}{2}\begin{bmatrix} 1 & 1-\sqrt{2} & 0 & 0 \end{bmatrix}^T,$$
$$q_2 = \frac{\sqrt{2+\sqrt{2}}}{2}\begin{bmatrix} 0 & 0 & 1 & 1-\sqrt{2} \end{bmatrix}^T$$
$$q_3 = \frac{\sqrt{2-\sqrt{2}}}{2}\begin{bmatrix} 1 & 1+\sqrt{2} & 0 & 0 \end{bmatrix}^T,$$
$$q_4 = \frac{\sqrt{2-\sqrt{2}}}{2}\begin{bmatrix} 0 & 0 & 1 & 1+\sqrt{2} \end{bmatrix}^T$$

decomposes the system into the uncoupled form.

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \xi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + (3-\sqrt{2}-\xi^2)\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0 \tag{26}$$

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \xi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + (3+\sqrt{2}-\xi^2)\begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = 0 \tag{27}$$

in which each uncoupled subsystem has just two-degrees-of-freedom. ■

*Remark 4.* Consider the relation

$$\tilde{K}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{K}$$

which we will be using shortly. As mentioned before, this is equivalent to the relation.

$$KG = GK \tag{28}$$

Furthermore, when $K = Q\Lambda Q^T$, $G = Q\Gamma Q^T$, and $Q$ is orthogonal (Eqs. (18) and (15)), Eq. (28) can be rewritten as

$$\Lambda\Gamma = \Gamma\Lambda \tag{29}$$

■

Corollary 1. *Let $\tilde{M} = \tilde{M}^T > 0$, $\tilde{K} = \tilde{K}^T$, $\tilde{G} = -\tilde{G}^T$ be n-by-n matrices and rank$(\tilde{G}) = 2m \le n$. A necessary and sufficient condition for there to exist a real change of coordinates that transforms Eq. (1) to the form given in Eq. (15) with*

$$\Lambda = \mathrm{diag}(\lambda_1 I_2, \lambda_2 I_2, \ldots, \lambda_m I_2, \lambda_{2m+1}, \lambda_{2m+2}, \ldots, \lambda_n) \tag{30}$$

*where each diagonal block is proportional to the 2 by 2 identity matrix $I_2$, and*

$$\Gamma = \mathrm{diag}\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \ldots \quad \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad 0, \quad \ldots \quad 0\right) \tag{31}$$

*is*

$$\tilde{K}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{K} \tag{32}$$

*Proof.* Sufficiency: Suppose that $\tilde{K}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{K}$. Then, the condition given in Eq. (21) is satisfied, since

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K} \tag{33}$$

Also, the condition given in Eq. (22), which is $(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})^2 = (\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2$, is obviously satisfied when $\tilde{K}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{K}$.

Thus, according to *Result 2*, there exists a linear transformation which transforms Eq. (1) to the form given in Eq. (15). Moreover from Remark 4, Eq. (32) implies that $\Lambda\Gamma = \Gamma\Lambda$ which becomes

$$\mathrm{diag}\left(\beta_1\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_2 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & \lambda_{2m-1} \\ -\lambda_{2m} & 0 \end{bmatrix}, 0, \ldots, 0\right) =$$

$$\mathrm{diag}\left(\beta_1\begin{bmatrix} 0 & \lambda_2 \\ -\lambda_1 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & \lambda_{2m} \\ -\lambda_{2m-1} & 0 \end{bmatrix}, 0, \ldots, 0\right)$$

From this, it follows that $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, $\cdots$, $\lambda_{2m-1} = \lambda_{2m}$, because $\beta_j \neq 0$.

Necessity: Suppose a linear coordinate change $q = Pp$ exists (nonsingular $P$) such that Eq. (1) is transformed to Eq. (15) and $P^T\tilde{M}P = I$, $P^T\tilde{K}P = \Lambda$, and $P^T\tilde{G}P = \Gamma$, with $\Lambda$ and $\Gamma$ given in Eqs. (30) and (31). Then, $\tilde{M}^{-1} = PP^T$, $\tilde{G} = P^{-T}\Gamma P^{-1}$, $\tilde{K} = P^{-T}\Lambda P^{-1}$, so that $\tilde{K}\tilde{M}^{-1}\tilde{G} = P^{-T}\Lambda\Gamma P^{-1} = P^{-T}\Gamma\Lambda P^{-1} = \tilde{G}\tilde{M}^{-1}\tilde{K}$ since $\Lambda$ and $\Gamma$ now commute. Thus, if there is a real congruence transformation which transforms Eq. (1) to the form (Eq. (15)), with $\Lambda$ and $\Gamma$ given in Eqs. (30) and (31), then condition (Eq. (32)) is satisfied. ■

*Remark 5.* Working with the system described by Eq. (10), as shown in Remark 4, Eq. (32) can be rewritten as

$$KG = GK \tag{34}$$

The commutation of $K$ and $G$ ensures that the conditions in Eqs. (13) and (14), namely, $KG^2 = G^2K$ and $(KG)^2 = (GK)^2$, are satisfied.

Postmultiplication of Eq. (34) by $G$ gives

$$KG^2 = GKG = GGK = G^2K$$

where the second and third equalities follow from Eq. (34). We thus see that $\tilde{K}\tilde{M}^{-1}\tilde{G}=\tilde{G}\tilde{M}^{-1}\tilde{K} \Rightarrow KG^2=G^2K$, and, of course, $(KG)^2=(GK)^2$. ∎

*Remark 6.* We note that Corollary 1 gives a commutation condition analogous to the Caughey–O'Kelly necessary and sufficient condition, and the condition in Eq. (23) is obtained by simply replacing $\tilde{G}$ in Eq. (32) by $\tilde{D}$. This becomes more evident when we observe that in our case the two commuting matrices are $K$ and $G$; in the Caughey–O'Kelly result, the matrices are the symmetric matrices $K$ and $D$. However, the two analogous results are subtly different. This is because the matrix $\Lambda$ in Eq. (30) has a specific restrictive structure in which the $2m$ eigenvalues of $K$ appear in two-dimensional diagonal blocks, each proportional to the identity matrix, as seen in Eq. (30). From this we observe that, if $K$ has more than $(n-2m)$ distinct eigenvalues of multiplicity 1, commutation of $K$ and $G$ is not possible. ∎

*Remark 7.* It should be pointed out that it is well known that a sufficient condition for two real normal matrices (which $K$ and $G$ are) to be simultaneously quasi-diagonalized by a real orthogonal matrix $Q$ is that the two matrices commute [4]. What we have shown here is a proof which shows that the commutation condition for simultaneously reducing a symmetric matrix and a skew-symmetric matrix to the form shown in Eqs. (30) and (31) is both necessary and sufficient. It has the advantage of providing additional insight into the restricted form (see Eq. (30)) that such a simultaneous quasi-diagonalization results in when the commutation condition is satisfied. ∎

*Remark 8.* The conditions provided in Result 1 go beyond the necessary and sufficient condition obtained in Corollary 1, since the condition in Corollary 1 places restrictions on the eigenvalues of the matrix $K$. Thus, there exist matrix pairs $\{G, K\}$ that do *not* commute but can still be quasi-diagonalized. That is, the set containing matrix pairs $\{G, K\}$ that can be simultaneously quasi-diagonalized by an orthogonal transformation is, in general, larger that the set containing matrix pairs $\{G, K\}$ that commute, as illustrated by the following two simple examples. ∎

*Example 2.* Consider the four-degree-of-freedom system described by Eq. (1) in which

$$\tilde{M}=I_4,\ \tilde{K}=\mathrm{diag}(\lambda_1,\ \lambda_1,\ \lambda_2,\ \lambda_2),\ \lambda_1\neq\lambda_2,\ \lambda_1,\ \lambda_2\neq 0$$

and

$$\tilde{G}=\mathrm{diag}\left(\beta\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \beta\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right),\quad \beta\neq 0$$

where $I_4$ is the 4-by-4 identity matrix. Thus, $K=\tilde{K}$ and $G=\tilde{G}$. The matrices $\tilde{K}(K)$ and $\tilde{G}(G)$ commute, and by Corollary 1, this commutation condition guarantees an orthogonal matrix $Q$ to exist for the simultaneous quasi-diagonalization of $\tilde{K}(K)$ and $\tilde{G}(G)$. In this trivial example, the matrix $Q$ is obviously $I_4$.

Were we to replace the matrix $\tilde{K}(K)$ instead by the matrix $\tilde{K}_1=\mathrm{diag}(\lambda_1,\ \lambda_2,\ \lambda_2,\ \lambda_2)$ with $\lambda_1\neq\lambda_2,\ \lambda_1,\ \lambda_2\neq 0$, then the matrices $\tilde{K}_1(K_1)$ and $\tilde{G}(G)$ no longer commute. Notice also that $\tilde{K}_1(K_1)$ is not in form given in Eq. (30). By Corollary 1 (and the result in Ref. [4]), one is no longer guaranteed that a matrix $Q$ exists that leads to simultaneous quasi-diagonalization of $\tilde{K}_1(K_1)$ and $G$. However, Result 1 guarantees that such a matrix $Q$ does exist since the conditions given in Eqs. (13) and (14) are satisfied. This is seen by a simple computation, which shows that

$$\tilde{K}_1\tilde{G}^2=-\beta^2\mathrm{diag}(\lambda_1,\ \lambda_2,\ \lambda_2,\ \lambda_2)=\tilde{G}^2\tilde{K}_1$$

and

$$(\tilde{K}_1\tilde{G})^2=-\beta^2\mathrm{diag}(\lambda_1\lambda_2 I_2,\ \lambda_2^2 I_2)=(\tilde{G}\tilde{K}_1)^2$$

The matrix $Q$ simultaneously quasi-diagonalizes $\tilde{K}_1$ and $\tilde{G}$ is again, obviously, just the matrix $I_4$. ∎

*Example 3.* We consider next a less trivial example. Consider the dynamical system described by Eq. (10) in which the matrices

$$K=\begin{bmatrix}2 & 0 & 0 & 0 & 0 & -1\\ 0 & 3 & 0 & 0 & -1 & 0\\ 0 & 0 & 5.5 & -0.5 & 0 & 0\\ 0 & 0 & -0.5 & 5.5 & 0 & 0\\ 0 & -1 & 0 & 0 & 3 & 0\\ -1 & 0 & 0 & 0 & 0 & 2\end{bmatrix},$$

$$G=\begin{bmatrix}0 & 1.5 & 0 & 0 & -0.5 & 0\\ -1.5 & 0 & 0 & 0 & 0 & 0.5\\ 0 & 0 & 0 & 3 & 0 & 0\\ 0 & 0 & -3 & 0 & 0 & 0\\ 0.5 & 0 & 0 & 0 & 0 & -1.5\\ 0 & -0.5 & 0 & 0 & 1.5 & 0\end{bmatrix}$$

Their spectra are $\{1, 2, 3, 4, 5, 6\}$ and $\{\pm i, \pm 2i, \pm 3i\}$, respectively. The real canonical form of $G$ is therefore

$$\Gamma=\mathrm{diag}\left(1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right), 2\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, 3\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

and $rank(G)=6$. Since the eigenvalues of $K$ are distinct, the commutation of these two matrices is indeed ruled out. In fact, for the two matrices to commute, we would require that the spectrum of $K$ be restricted to $\{\lambda_1 I_2,\ \lambda_2 I_2,\ \lambda_3 I_2\}$ as seen in Corollary 1. While such multiple eigenvalues can arise in stiffness matrices in aerospace and mechanical engineering, their occurrence is generally not common.

Yet, simultaneous quasi-diagonalization of $K$ and $G$ is guaranteed since $KG^2=GK^2$ and $(KG)^2=(GK)^2$, as can be verified by straightforward computation. ∎

Another way to understand the strong restriction placed on the potential matrix, which the commutation condition given in Eq. (32) demands, is to ask the question: given a gyroscopic matrix, what are all the possible potential matrices that commute with it? How do they compare in number with all the possible potential matrices that satisfy the necessary and sufficient conditions for quasi-diagonalization given in Eqs. (21) and (22)? To illustrate this approach, we provide the following example.

*Example 4.* Assume that we have a four-degree-of-freedom system described by Eq. (10), with a gyroscopic matrix $G$ that is already in the canonical form given by

$$G=\mathrm{diag}\left(\beta_1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \beta_2\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right), \beta_1\neq\beta_2\neq 0$$

Our aim is to determine all the symmetric matrices $K$ that commute with this given matrix $G$. A symmetric 4-by-4 $K$ matrix is, of course, defined by 10 elements (parameters), and therefore, these elements constitute a 10-dimensional parameter space.

It is easy to see that all the possible $K$ matrices that commute with $G$ must have the structure

$$K=\mathrm{diag}(a_1 I_2,\ a_2 I_2)$$

thereby restricting the elements of $K$ to a 2-dimensional subspace from the 10-dimension parameter space of $K$.

Next, we look at those symmetric matrices $K_1$ that satisfy the twin necessary and sufficient conditions $KG^2=G^2K$ and $(KG)^2=(GK)^2$.

The structure of the matrix $K_1$ that satisfies both these conditions is

$$K_1 = K_1(a_1, \ldots, a_6) = \begin{bmatrix} a_1 & a_5 & 0 & 0 \\ a_5 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & a_6 \\ 0 & 0 & a_6 & a_4 \end{bmatrix}$$

where the parameters $a_i$, $i = 1, \ldots, 6$, are arbitrary. The twin necessary and sufficient conditions are thus seen to restrict the elements of $K_1$ to a six-parameter subspace of the 10-dimensional parameter space. Thus, the dimension of the subspace of parameters present in $K_1$ exceeds the dimension of the subspace of parameters present in $K$ by 4. This shows that the restriction imposed on the stiffness matrix by the condition of commutation is far greater than that posed on it by the necessary and sufficient conditions given in Eqs. (13) and (14). ∎

Examples 2–4 illustrate that the necessary and sufficient condition provided in Corollary 1 is strongly restrictive, and therefore less valuable, than the necessary and sufficient conditions given in Eqs. (13) and (14) for simultaneous quasi-diagonalization of $K$ and $G$.

The generalized coordinates $q$ that we are concerned with in this paper are real, and as mentioned in the Introduction, there are no real coordinate transformations that would completely decompose (diagonalize) the system described in Eq. (1). However, condition (Eq. (32)) can be shown to be necessary and sufficient for the complete decomposition of this system by means of a complex linear transformation (*congruence). Indeed, according to a well-known result [4], the two Hermitian matrices $iG$ and $K$ ($(iG)^* = -iG^T = iG$, $K^* = K^T = K$) can be simultaneously diagonalized by a unitary matrix $U$ if and only if $iG$ and $K$ commute. Here, $(\ )^*$ is a complex conjugate transpose operators. The fact that $U$ is unitary means that $U^*U = I$ and, consequently, whenever the matrices $iG$ and $K$ commute, they possess a common system of $n$ mutually orthogonal eigenvectors—the columns of matrix $U$. Create the unitary matrix $U$ as

$$U = \begin{bmatrix} u_1 & \bar{u}_1 & \ldots & u_m & \bar{u}_m & u_{2m+1} & \ldots & u_n \end{bmatrix}$$

where the complex conjugate vectors $u_j$ and $\bar{u}_j$ correspond to pairs of nonzero eigenvalues of the matrix $G$, $\mp i\beta_j$, $j = 1, \ldots, m$, and the real vectors $u_{2m+1}, \ldots, u_n$ correspond to the zero eigenvalue of $G$. Noting that a pair of conjugate complex eigenvectors of the symmetric matrix $K$ corresponds to a pair of its real *equal* eigenvalues, we have $U^*iGU = \mathrm{diag}(\beta_1, -\beta_1, \ldots, \beta_m, -\beta_m, 0, \ldots, 0)$ and $U^*KU = \mathrm{diag}(\lambda_1, \lambda_1, \ldots, \lambda_m, \lambda_m, \lambda_{2m+1}, \ldots, \lambda_n)$. Multiplying Eq. (1) from the left by $U^*\tilde{M}^{-1/2}$ and using the transformation $q = \tilde{M}^{-1/2}Uz$, we obtain the following 2 m uncoupled complex conjugate equations

$$\left.\begin{aligned} \ddot{z}_j - i\beta_j\dot{z}_j + \lambda_j z_j = 0 \\ \ddot{\bar{z}}_j + i\beta_j\dot{\bar{z}}_j + \lambda_j \bar{z}_j = 0 \end{aligned}\right\} \quad j = 1, \ldots, m$$

and ($n$−2 m) real ones

$$\ddot{z}_j + \lambda_j z_j = 0, \ j = 2m+1, \ldots, n$$

Although this system is completely uncoupled in the complex $z$ variables, it should be said that in real coordinates, each pair of conjugate complex equations corresponds to a gyroscopically coupled system with two-degrees-of-freedom. More precisely, separating the real and imaginary parts of the variables $z_j$ ($j = 1, \ldots, m$) as $z_j = \xi_j + i\eta_j$, we obtain

$$\begin{bmatrix} \ddot{\xi}_j \\ \ddot{\eta}_j \end{bmatrix} + \beta_j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\xi}_j \\ \dot{\eta}_j \end{bmatrix} + \lambda_j \begin{bmatrix} \xi_j \\ \eta_j \end{bmatrix} = 0$$

which, of course, agrees with Eqs. (30) and (31).

We next particularize the result in Result 2 to skew-symmetric matrices $\tilde{G}$ whose nonzero eigenvalues are distinct.

**Result 3**

Let $\tilde{M} = \tilde{M}^T > 0$, $\tilde{K} = \tilde{K}^T$, $\tilde{G} = -\tilde{G}^T$ and $rank(\tilde{G}) = 2m \le n$. If all the nonzero eigenvalues of the matrix $\tilde{M}^{-1}\tilde{G}(G)$ are distinct, then the necessary and sufficient condition for the existence of a real linear change of coordinates that transforms Eq. (1) to the form given in Eqs. (15)–(17) is

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}$$

which is the condition given in Eq. (21).

*Proof.* Noting that $\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K} \Leftrightarrow KG^2 = G^2K$, we can consider the equivalent system described by Eq. (10). It therefore suffices then to show that, under the assumptions given, that $KG^2 = G^2K \twoheadrightarrow (KG)^2 = (GK)^2$.

According to Lemma 1, there exists a real orthogonal matrix $Q$ such that

$$G = Q\begin{bmatrix} \hat{G} & 0 \\ 0 & 0_{n-2m} \end{bmatrix}Q^T \tag{35}$$

and

$$K = Q\begin{bmatrix} \hat{K} & \bar{K} \\ \bar{K}^T & \hat{\hat{K}} \end{bmatrix}Q^T \tag{36}$$

where

$$\hat{G} = \mathrm{diag}\left(\beta_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \tag{37}$$

with $\beta_j \neq 0$, $(j = 1, \ldots, m)$, and from the statement of Result 3, $\beta_j \neq \beta_k$ for $j \neq k$. The matrix $0_{n-2m}$ is an $(n-2m)$-dimensional zero matrix, $\hat{K}$ and $\hat{\hat{K}}$ are $2m$ and $(n-2m)$ dimensional symmetric matrices respectively, and $\bar{K}$ is a $2m$ by $(n-2m)$ matrix. Equation (13) then yields $\bar{K} = 0$ because $\hat{G}^2 = -\mathrm{diag}(\beta_1^2 I_2, \ldots, \beta_m^2 I_2)$ is nonsingular, and

$$\hat{K}\hat{G}^2 = \hat{G}^2\hat{K} \tag{38}$$

Next, after partitioning the symmetric matrix $\hat{K}$, as $\hat{K} = [\hat{K}_{jk}]_{j,k=1}^m$ with two-dimensional submatrices $\hat{K}_{ij}$, Eq. (38) becomes

$$\beta_j^2\hat{K}_{jk} = \beta_k^2\hat{K}_{jk}, \ j, k = 1, \ldots, m \tag{39}$$

For $j \neq k$, Eq. (39) becomes $(\beta_j^2 - \beta_k^2)\hat{K}_{jk} = 0$, and since $\beta_j \neq \beta_k$ for $j \neq k$, we find that $\hat{K}_{jk} = 0$ for $j \neq k$. Thus, the matrix $\hat{K}$ is block diagonal with two-dimensional submatrices $K_{jj}$ along its diagonal, i.e., $\hat{K} = \mathrm{diag}(\hat{K}_{11}, \ldots, \hat{K}_{mm})$. Denoting this block-diagonal matrix by $\hat{K} := \mathrm{diag}(\hat{K}_{jj})_{j=1}^m$, we thus find that the matrix $K$ that satisfies Eq. (13) must have the form

$$K = Q\begin{bmatrix} \mathrm{diag}(\hat{K}_{jj})_{j=1}^m & 0 \\ 0 & \hat{\hat{K}} \end{bmatrix}Q^T \tag{40}$$

where $\hat{\hat{K}}$ is an $(n-2m)$-dimensional symmetric matrix. Now, it follows that $(KG)^2 = (GK)^2$, where the matrices $G$ and $K$ have their structures are shown in Eqs. (35) and (37) and in Eq. (40). This can be seen from the two-dimensional matrices

$$\hat{K}_{jj} = \begin{bmatrix} k_{a,jj}k_{b,jj} \\ k_{b,jj}k_{c,jj} \end{bmatrix}, \hat{\beta} := \beta_j\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

that satisfy this condition since

$$(\hat{K}_{jj}\hat{\beta})^2 = \beta_j^2\begin{bmatrix} k_{b,jj}^2 - k_{a,jj}k_{c,jj} & 0 \\ 0 & k_{b,jj}^2 - k_{a,jj}k_{c,jj} \end{bmatrix} = (\hat{\beta}\hat{K}_{jj})^2$$

Then, Result 3 follows from Result 2. ∎

*Remark 9.* Result 3 gives a single necessary and sufficient (n&s) condition that guarantees simultaneous quasi-diagonalization of $K$ and $G$ just like the analogous single n&s condition of Caughey–O'Kelly. However, Result 3 applies only to skew-symmetric matrices whose nonzero eigenvalues are distinct. Moreover, the n&s conditions for the simultaneous quasi-diagonalization of $K$ and $G$ and those for the analogous simultaneous diagonalization of $K$ and (symmetric) $D$ are different in character. For simultaneous quasi-diagonalization, the n&s condition is $KG^2 = G^2K$ while for Caughey–O'Kelly's simultaneous diagonalization it is $KD = DK$. Note that $KG^2 = G^2K \not\Rightarrow KG = GK$ (see Example 4).

Referring back to Examples 3 and 4 in which $\beta_j$ are all distinct, the condition $(KG)^2 = (GK)^2$ is automatically satisfied since $KG^2 = GK^2$. ■

Corollary 2. *Let $\tilde{M} = \tilde{M}^T > 0$, $\tilde{K} = \tilde{K}^T$, $\tilde{G} = -\tilde{G}^T \neq 0$. If $rank(\tilde{G}) = 2$, then condition (Eq. (21)) is a necessary and sufficient condition for Eq. (1) to be transformed to Eqs. (15)–(17) using a real linear change of coordinates.*

*Proof.* When $rank(\tilde{G}) = 2$, by Lemma 1 the nonzero eigenvalues of $\tilde{G}$ have got to be distinct. Result 3 is therefore applicable. ■

We illustrate Result 3 and its corollary by the following example.

*Example 5.* Consider the system described by Eq. (1) with

$$\tilde{M} = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}, \tilde{G} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}, \tilde{K} = \begin{bmatrix} 13 & -3 & 14 \\ -3 & 10 & 3 \\ 14 & 3 & 13 \end{bmatrix} \tag{41}$$

We note that since $\tilde{G}$ has dimension 3, which is an odd number, one of its eigenvalues is zero, and the other two must be distinct since they have to be of the form $\pm i\beta$.

First, we calculate

$$\tilde{M}^{-1} = \frac{1}{9}\begin{bmatrix} 5 & 0 & -4 \\ 0 & 2.25 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

and

$$\tilde{K}\tilde{M}^{-1}\tilde{G} = \begin{bmatrix} 1.5 & -2 & -1.5 \\ -5 & -12 & 5 \\ -1.5 & 2 & 1.5 \end{bmatrix}$$

Since the matrix $\tilde{K}\tilde{M}^{-1}\tilde{G}$ is not skew-symmetric (i e., $\tilde{K}\tilde{M}^{-1}\tilde{G} \neq \tilde{G}\tilde{M}^{-1}\tilde{K}$), Corollary 1 is not applicable. However,

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \begin{bmatrix} 1 & 6 & -1 \\ 6 & -20 & -6 \\ -1 & -6 & 1 \end{bmatrix} = (\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G})^T$$
$$= \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}$$

and according to Result 3, there exist principal coordinates so that the system decomposes into one two-degree and one single-degree-of-freedom subsystem. In order to obtain the transformation matrix $P$ that decomposes the system, we look for the solution of the generalized symmetric eigenvalue problem $\tilde{K}u = \lambda\tilde{M}u$. We get the eigenvalues and corresponding eigenvectors normalized with respect to the mass matrix $\tilde{M}$, as follows:

$$\lambda_1 = -2, q_1 = \frac{1}{\sqrt{22}}\begin{bmatrix} 3 & 1 & -3 \end{bmatrix}^T; \lambda_2 = 3, q_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T;$$

$$\lambda_3 = \frac{7}{2}, q_3 = \frac{1}{\sqrt{11}}\begin{bmatrix} 1 & -\frac{3}{2} & -1 \end{bmatrix}^T$$

Since $\tilde{G}q_2 = 0$, we introduce principal coordinates $p(t) = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^T$ by the transformation $q(t) = Pp(t)$, where $P = \begin{bmatrix} q_1 & q_3 & q_2 \end{bmatrix}$. Now, it is easy to verify that this transformation reduces Eq. (1) with coefficient matrices as in Eq. (41) to the form

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} -2p_1 \\ 3.5p_2 \end{bmatrix} = 0$$

$$\ddot{p}_3 + 3p_3 = 0$$ ■

In the general case, the condition given in Eq. (13) (or Eq. (21)) does not imply the condition given in Eq. (14) (or Eq. (22)) and vice versa, as the following two numerical examples show.

*Example 6.* Consider the system described by Eq. (10) with

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{42}$$

For this example, we have $G^2 = -I$, and obviously, condition (Eq. (13)) is satisfied. However, the matrix

$$(KG)^2 = \begin{bmatrix} -3 & 0 & 0 & -2 \\ 0 & -3 & 1 & 2 \\ 2 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

is asymmetric, i.e., $(KG)^2 \neq (GK)^2$, and therefore, the system *cannot* be decomposed into two independent subsystems.

To verify this, we solve the eigenvalue problem for $K$ and obtain

$$\lambda_1 = 0.1206, q_1 = \begin{bmatrix} 0.228 & 0.4285 & 0.5774 & 0.6565 \end{bmatrix}^T$$

$$\lambda_2 = 1, q_2 = 0.5774\begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix}^T$$

$$\lambda_3 = 2.3473, q_3 = \begin{bmatrix} 0.6565 & -0.228 & -0.5774 & 0.4285 \end{bmatrix}^T$$

and

$$\lambda_4 = 3.5321, q_4 = \begin{bmatrix} -0.4285 & 0.6565 & -05774 & 0.228 \end{bmatrix}^T$$

Substitution of $x = Qp$ into Eq. (10), with $Q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}$ and premultiplying by $Q^T$, yields

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \\ \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + \begin{bmatrix} 0 & -0.4492 & 0.2932 & 0.844 \\ 0.4492 & 0 & -0.8441 & 0.2931 \\ -0.2932 & 0.8441 & 0 & 0.4491 \\ -0.844 & -0.2931 & -0.4491 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + \begin{bmatrix} 0.1206p_1 \\ p_2 \\ 2.3473p_3 \\ 3.5321p_4 \end{bmatrix} = 0$$

and as we see in the principal coordinates, the system does not decompose. ■

*Example 7.* Consider the system described by Eq. (10) with

$$G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

It is clear that $G^2 = -\text{diag}(1, 1, 0)$ and

$$KG^2 = -\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

so $KG^2$ is asymmetric and the condition in Eq. (13) is *not* satisfied. On the other hand, the matrix $(KG)^2 = 0 = (GK)^2$, and consequently, the condition in Eq. (14) is satisfied. Note that in the coordinates $p = [p_1 \quad p_2 \quad p_3]^T$ obtained by the transformation using the modal matrix

$$P = \begin{bmatrix} 0.8944 & 0.2433 & 0.3753 \\ -0.4472 & 0.4865 & 0.7506 \\ 0 & -0.8391 & 0.5439 \end{bmatrix}$$

the transformed system has the coupled form with respect to velocities given by

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \\ \ddot{p}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0.5439 & 0.8392 \\ -0.5439 & 0 & 0 \\ -0.8392 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.5505 p_2 \\ 6.4495 p_3 \end{bmatrix} = 0$$

■

## 4 Decomposition and Uncoupling of Gyroscopic Systems With Positive Definite Stiffness Matrices

In many applications in science and engineering, the potential (stiffness) matrix of the system is positive definite. In this section, we consider linear gyroscopic systems whose mass matrices, $\tilde{M}$, are symmetric and whose stiffness matrices, $\tilde{K}$, are positive definite. Since $\tilde{K}^{1/2}$ is now uniquely defined, premultiplying Eq. (1) by $\tilde{K}^{-1/2}$ we obtain

$$M\ddot{y} + G\dot{y} + y = 0 \tag{43}$$

where the symmetric matrix

$$M = \tilde{K}^{-1/2}\tilde{M}\tilde{K}^{-1/2} \tag{44}$$

the skew-symmetric matrix

$$G = \tilde{K}^{-1/2}\tilde{G}\tilde{K}^{-1/2} \tag{45}$$

and

$$q = \tilde{K}^{-1/2}y$$

*Remark 10.* Observe that Eq. (43) is analogous to Eq. (10), and the right-hand sides of Eqs. (44) and (45) are obtained from Eqs. (11) and (12) in Sec. 3 by simply interchanging the symbols $\tilde{M}$ and $\tilde{K}$. Instead of aiming to quasi-diagonalize the matrices $G$ and $K$ in Eq. (10), as we did in Sec. 3, we must now quasi-diagonalize $G$ and $M$ in Eq. (43) instead. ■

**Result 4**
Consider the system described in Eq. (43) in which the skew-symmetric matrix $G$ has rank $2m \le n$. Then, the necessary and sufficient conditions for Eq. (43) to be decomposed by an orthogonal congruence transformation into uncoupled subsystems, $m$ of which are two-degrees-of-freedom and $n-2m$ of which are single-degree-of-freedom subsystems, is that

$$MG^2 = G^2M \tag{46}$$

and

$$(MG)^2 = (GM)^2 \tag{47}$$

The uncoupled equations in the principal coordinates $p$ have the form

$$R\ddot{p} + \Gamma p + p = 0 \tag{48}$$

with

$$R = \text{diag}(\rho_1, \rho_2, \ldots, \rho_n) \tag{49}$$

and

$$\Gamma = \text{diag}\left(\gamma_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \ldots \quad \gamma_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad 0, \quad \ldots \quad 0\right) \tag{50}$$

where $\rho_j$ and $\gamma_j$ are real numbers.

*Proof.* In view of Remark 10, all we need to do to get Eqs. (46) and (47) is simply replace the $K$ in Eqs. (13) and (14) by $M$. Equations (48)–(50) are obtained in a manner analogous to those obtained before in Result 1 of Sec. 3. ■

The real numbers $\rho_j$ $(j = 1, \ldots, n)$ in Eq. (49) are eigenvalues of the matrix $\tilde{K}^{-1}\tilde{M}(M)$, and the real numbers $\gamma_j$ $(j = 1, \ldots, m)$ in Eq. (50) indicate that the nonzero eigenvalues of $\tilde{K}^{-1}\tilde{G}(G)$ are $\pm i\gamma_j$ $(j = 1,\ldots,m)$.

**Result 5**
Let $\tilde{M} = \tilde{M}^T$, $\tilde{K} = \tilde{K}^T > 0$, $\tilde{G} = -\tilde{G}^T$ and $rank(\tilde{G}) = 2m \le n$. The necessary and sufficient conditions for Eq. (1) to be transformed by a linear coordinate change to the form given in Eqs. (48)–(50) are

$$\tilde{M}\tilde{K}^{-1}\tilde{G}\tilde{K}^{-1}\tilde{G} = \tilde{G}\tilde{K}^{-1}\tilde{G}\tilde{K}^{-1}\tilde{M} \tag{51}$$

and

$$(\tilde{M}\tilde{K}^{-1}\tilde{G}\tilde{K}^{-1})^2 = (\tilde{G}\tilde{K}^{-1}\tilde{M}\tilde{K}^{-1})^2 \tag{52}$$

*Proof.* In view of Remark 10, all we need to do is interchange the symbols $\tilde{M}$ and $\tilde{K}$ in Eqs. (21) and (22) in Sec. 3 to get Eqs. (51) and (52). ■

As explained in Remark 10, the exchange $M(\tilde{M}) \leftrightarrow K(\tilde{K})$ then permits us to obtain results for systems described by Eqs. (43) (or Eq. (1)) in which $K(\tilde{K})$ is positive definite and $M(\tilde{M})$ is symmetric that are analogous to those already obtained before in Sec. 3 for the systems described by Eq. (10) (or Eq. (1)) in which $M(\tilde{M})$ is positive definite and $K(\tilde{K})$ is symmetric. We provide just one such example below, which is analogous to Corollary 1 in Sec. 3.

Corollary 3. *Let* $\tilde{M} = \tilde{M}^T$, $\tilde{K} = \tilde{K}^T > 0$, $\tilde{G} = -\tilde{G}^T$ *and* $rank(\tilde{G}) = 2m$. *A necessary and sufficient condition for there to exist a real linear change of coordinates that transforms Eq. (1) to the form given in Eq. (48) with*

$$R = \text{diag}(\rho_1 I_2, \rho_2 I_2, \ldots, \rho_m I_2, \rho_{2m+1}, \rho_{2m+2}, \ldots, \rho_n) \tag{53}$$

and

$$\Gamma = \text{diag}\left(\gamma_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \ldots \quad \gamma_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad 0, \quad \ldots \quad 0\right) \tag{54}$$

is

$$\tilde{M}\tilde{K}^{-1}\tilde{G} = \tilde{G}\tilde{K}^{-1}\tilde{M} \tag{55}$$

*Remark 11.* If $\tilde{M} = \tilde{M}^T > 0$ and $\tilde{K} = \tilde{K}^T > 0$, then Result 5 (Corollary 3) is equivalent to Result 2 (Corollary 1). Also, if the potential matrix $\tilde{K}$ is negative definite Result 5 and its corollary can be applied after the premultiplication of Eq. (1) by $-1$. ■

*Remark 12.* Condition (55) of Corollary 3 is necessary and sufficient for the diagonalization of the system in complex coordinates using a linear complex coordinate change. ■

## 5 Conclusions

This paper deals with multi-degree-of-freedom linear gyroscopic potential systems. Such systems are found in nature and in engineered systems, especially in the areas of aerospace and mechanical engineering. The paper relies on a recent result related to the development of the two necessary and sufficient conditions for simultaneous quasi-diagonalization of a skew-symmetric and a symmetric matrix by means of an orthogonal congruence. This is analogous to the well-known necessary and sufficient condition used by Caughey–O'Kelly for the simultaneous diagonalization of two symmetric matrices.

The analogy between a linear multi-degree-of-freedom damped potential system that has a symmetric damping matrix $\tilde{D}$ and a linear multi-degree-of-freedom gyroscopic potential system that has a skew-symmetric matrix $\tilde{G}$ stems from the fact that the former system turns into the latter by replacing the matrix $\tilde{D}$ in its equation of motion by the matrix $\tilde{G}$. Because of this formal analogy, it is useful to compare results obtained for uncoupling gyroscopic systems with the well-known results obtained for the uncoupling of damped potential systems.

However, the replacement $\tilde{D}$ by $\tilde{G}$ causes a considerable change in the system since the nonzero eigenvalues of $\tilde{G}$ are all conjugate imaginary pairs, while the eigenvalues of $\tilde{D}$ are all real. This precludes the decomposition of a gyroscopic potential system into uncoupled single-degree-of-freedom subsystems through the use of a *real* coordinate change. The best that can be done using a real coordinate change is to uncouple the system into subsystems each of which has at most two-degrees-of-freedom. This is what has been accomplished in this paper, and the necessary and sufficient conditions for decoupling a gyroscopic potential system are provided here. Such an uncoupling provides a new and deeper understanding of the behavior of multi-degree-of-freedom gyroscopic potential systems in terms of one- and two-degree-of-freedom subsystems that are much simpler and easier to understand.

Section 3 assumes that the mass matrix of the linear gyroscopic dynamical system is positive definite and the stiffness (potential) matrix is symmetric. We explore the use of real coordinate changes to convert any such multi-degree-of-freedom gyroscopic dynamical system into uncoupled subsystems. Below, we summarize some of the main qualitative findings obtained in this paper.

(1) An $n$-degree-of-freedom linear gyroscopic potential system, whose gyroscopic matrix has rank $2m \leq n$, can be decomposed by a suitable real linear change in coordinates into $m$ uncoupled simple, two-degree-of-freedom subsystems and $(n-2m)$ simple, one-degree-of-freedom subsystems if and only if the twin conditions obtained in the paper are satisfied. It should be noted that *two independent* necessary and sufficient conditions for such a decoupling need to be *simultaneously* satisfied.

These uncoupled two-degree-of-freedom subsystems are each gyroscopic potential subsystems in canonical form, and the uncoupled single-degree-of-freedom subsystems are each pure potential systems, also in canonical form.

(2) The uncoupling to at most two-degree-of-freedom subsystems improves our fundamental understanding of the complex behavior of multi-degree-of-freedom gyroscopic potential systems, much like the Caughey–O'Kelly uncoupling of multi-degree-of-freedom viscously damped vibrating systems does.

(3) The results obtained have considerable computational value since one only needs to compute the responses of uncoupled two-degree-of-freedom gyroscopic potential subsystems and single-degree-of-freedom pure potential subsystems, both of which can be obtained in closed form. This makes the determination of the response of large-scale multi-degree-of-freedom gyroscopic systems to impressed external forces computationally more efficient, while providing greater insights into the computed results.

(4) A commutation condition for a gyroscopic potential system (which is analogous to the Caughey–O'Kelly commutation condition for a damped potential system) provides a necessary and sufficient condition for the gyroscopic system to be uncoupled in the manner described in (1) with the further restriction that each of the $m$ two-degree-of-freedom gyroscopic subsystems has potential matrices with *double* eigenvalues (double frequencies of vibration). Such a decomposition is also, as we have shown, a real analogue of the complete decoupling (diagonalization) of gyroscopic systems by complex coordinate changes.

From a practical viewpoint, this restriction that is imposed by the commutation condition on the nature of the stiffness (potential) matrix is quite severe, since the requirement that the stiffness matrix must have several eigenvalues each with multiplicity greater than 1 is not commonly observed in both naturally occurring as well as engineered structures. Therefore, the practical usefulness of this commutation condition appears limited.

(5) Because this commutation condition for multi-degree-of-freedom gyroscopic potential systems is restrictive, given a mass matrix, the number of (potential matrix, skew-symmetric matrix) pairs that satisfy the twin necessary and sufficient (n&s) conditions provided here—which guarantee decomposition to uncoupled subsystems (as described in (1) above)—, far exceed, roughly speaking, the number of such pairs that satisfy the commutation condition described in item (4) above.

Being less restrictive, the twin necessary and sufficient conditions obtained here are therefore more valuable in determining whether a multi-degree-of-freedom gyroscopic system can be decomposed into uncoupled subsystems, each with at most two-degrees-of-freedom.

(6) If the gyroscopic matrix that describes the gyroscopic potential system has distinct nonzero eigenvalues, then the two independent necessary and sufficient conditions obtained for decoupling the multi-degree-of-freedom gyroscopic system (in the manner described in (1)) reduce to just a single necessary and sufficient condition. However, this condition is not the gyroscopic analog of the Caughey–O'Kelly commutation condition.

In Sec. 4, all the results obtained in Sec. 3 are extended to linear multi-degree-of-freedom gyroscopic potential systems whose stiffness matrices are assumed to be positive definite. All the analytical results corresponding to those obtained in Sec. 3 are shown to be obtained for such systems by a simple interchange of symbols used in Sec. 3.

Several examples are considered throughout the paper illustrating various facets of the analytical results, thereby demonstrating the improvement in our fundamental understanding of the dynamics of multi-degree-of-freedom gyroscopic potential systems when the necessary and sufficient conditions described in the paper are satisfied.

## Conflict of Interest

There are no conflicts of interest.

## Data Availability Statement

No data, models, or code were generated or used for this paper.

## Appendix

**Proof of Lemma 2**. (a) *Necessity*. The necessity of the conditions of Lemma 2 is almost obvious. Indeed, if an orthogonal reduction to forms (Eqs. (5) and (6)) is possible, then the matrices $KG^2 = G^2K$ and $(KG)^2 = (GK)^2$, because they are orthogonally similar to the diagonal matrices $\Lambda\Gamma^2$ and $(\Lambda\Gamma)^2$. For example, assuming Eqs. (5) and (6) are true, we have $K = Q\Lambda Q^T$ and $G = Q\Gamma Q^T$, so that $KG^2 = Q\Lambda\Gamma^2 Q^T = Q\Gamma^2\Lambda Q^T = G^2K$ with the second equality in the chain following from Remark 1.

(b) *Sufficiency*. Since the matrix $G \neq 0$, let $spec(G) = (\pm i\beta_1,\ldots, \pm i\beta_m, 0,\ldots, 0)$, $\beta_j \neq 0$ $(j = 1, \ldots, m)$ be the spectrum of $G$. From Eq. (19), $spec(G^2) = (-\beta_1^2, -\beta_1^2, \ldots, -\beta_m^2, -\beta_m^2, 0, \ldots, 0)$. Suppose that conditions (Eqs. (7) and (8)) are satisfied. These two conditions ensure that the matrices $K$ and $G^2$ commute and that the matrices $K$ and $GKG$ commute. Furthermore, using Eq. (7), we find that $(G^2)(GKG) = GG^2KG = GKG^2G = (GKG)(G^2)$, which shows that $G^2$ and $GKG$ also commute. Hence, the symmetric matrices, $G^2$, $K$, and $GKG$ commute pairwise. This is the necessary and sufficient condition for a (real) orthogonal matrix to exist such that the three matrices can be simultaneously diagonalized; thus, the three matrices have a complete set of common eigenvectors. The columns of this orthogonal matrix can be reordered so that the eigenvalues of the diagonal matrix resulting from the diagonalization of $G^2$ (in the simultaneous diagonalization of the three matrices) can be placed in any desired order. Therefore, with no loss of generality, the first common unit eigenvector, $q_1$, of the orthogonal matrix, can be taken to be such that

$$G^2 q_1 = -\beta_1^2 q_1,\ \beta_1 \neq 0$$
$$K q_1 = \lambda_1 q_1$$
$$GKG q_1 = \mu_1 q_1$$

where $\lambda_1$ and $\mu_1$ are real numbers, which could also be zero.

Premultiplying the last equation by $G$ gives $G^2KGq_1 = \mu_1 G q_1$, and taking into account that $KG^2 = G^2K$, we get $KGG^2q_1 = \mu_1 G q_1$. Noting that $G^2 q_1 = -\beta_1^2 q_1$, we then obtain

$$K(Gq_1) = -\mu_1\beta_1^{-2}(Gq_1)$$

It therefore follows that $-Gq_1$ is also an eigenvector of $K$. Since $\|Gq_1\| = \sqrt{q_1^T G^T G q_1} = \sqrt{-q_1^T G^2 q_1} = \sqrt{\beta_1^2 q_1^T q_1} = \beta_1 \neq 0$, we see that the vector $-\beta_1^{-1}Gq_1$ is a unit eigenvector of $K$ corresponding to the eigenvalue $\lambda_2 = -\mu_1\beta_1^{-2}$. We shall denote it by $q_2 := -\beta_1^{-1}Gq_1$ in what follows. Furthermore, $G$ is skew-symmetric, and therefore, $q_2^T q_1 = \beta_1^{-1} q_1^T G q_1 = 0$; i.e., the unit vectors $q_1$ and $q_2$ are orthogonal. We therefore find that $q_1$ and $q_2$ are two different orthogonormal eigenvectors of $K$.

We now use these two orthonormal unit vectors, $q_1$ and $q_2$, as the first and second columns of the orthogonal matrix $Q_1 = \begin{bmatrix} q_1 & q_2 & q_3 & \ldots & q_n \end{bmatrix}$ in which the remaining columns are chosen arbitrarily, provided $Q_1^T Q_1 = I_n$. Our purpose is to determine the structure of the symmetric matrix $Q_1^T K Q_1 := [q_i^T K q_j]$ and the skew-symmetric matrix $Q_1^T G Q_1 := [q_i^T G q_j]$ by determining the first two rows (columns) of each of them.

We see that for $j = 1, 2, \ldots, n$, noting the orthogonality of the columns of $Q_1$, the elements of the first and second rows (columns) of $Q_1^T K Q_1$ are given, respectively, by

$$q_1^T K q_j = q_j^T K q_1 = \lambda_1 q_j^T q_1 = \lambda_1 \delta_{1j}$$

and

$$q_2^T K q_j = q_j^T K q_2 = \lambda_2 q_j^T q_2 = \lambda_2 \delta_{2j}$$

where $\delta_{ij}$ denotes the Kronecker delta. Likewise, for $j = 1, 2, \ldots, n$, noting that $Gq_1 = -\beta_1 q_2$ and $Gq_2 = -\beta_1^{-1} G^2 q_1 = \beta_1 q_1$, the elements of the first and second rows (columns) of $Q_1^T G Q_1$ are given, respectively, by the relations

$$q_1^T G q_j = -q_j^T G q_1 = \beta_1 q_j^T q_2 = \beta_1 \delta_{2j}$$

and

$$q_2^T G q_j = -q_j^T G q_2 = -\beta_1 q_j^T q_1 = -\beta_1 \delta_{1j}$$

We therefore observe that the matrices $Q_1^T K Q_1$ and $Q_1^T G Q_1$ have the following structure:

$$Q_1^T K Q_1 = \left[\begin{array}{cc:c} \lambda_1 & 0 & \\ 0 & \lambda_2 & 0 \\ \hdashline \multicolumn{2}{c:}{0} & K_{n-2} \end{array}\right]$$

and

$$Q_1^T G Q_1 = \left[\begin{array}{cc:c} 0 & \beta_1 & \\ -\beta_1 & 0 & 0 \\ \hdashline \multicolumn{2}{c:}{0} & G_{n-2} \end{array}\right]$$

Since the $(n-2)$-dimensional matrices $K_{n-2}$ and $G_{n-2}$ satisfy the same conditions as $K$ and $G$, this procedure continues in the same manner, and after $m$ steps, we conclude that there exists an orthogonal matrix $\tilde{Q}$ such that

$$\tilde{Q}^T G \tilde{Q} = \text{diag}\left(\beta_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \beta_m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right)$$

and

$$\tilde{Q}^T K \tilde{Q} = \text{diag}(\lambda_1, \ldots, \lambda_{2m}, K_{n-2m})$$

where $K_{n-2m}$ is an $(n-2m)$-dimensional symmetric matrix.

It remains to observe that there always exists an orthogonal matrix $Q_{n-2m}$ of order $(n-2m)$ that reduces the matrix $K_{n-2m}$ to diagonal form and, consequently, the orthogonal matrix

$$Q = \tilde{Q} \begin{bmatrix} I_{2m} & 0 \\ 0 & Q_{n-2m} \end{bmatrix}$$

reduces $K$ and $G$ to the forms given in Eqs. (5) and (6).

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